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On the symmetries and invariants of the harmonic oscillator

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Abstract. A general formulation of Noether's theorem is applied to the equation of a harmonic oscillator. The definition of symmetry includes the usual Lie invariance as a special case and (unlike standard formulations) generates the full set of invariants (i.e. gives closure under functional composition). The analysis for a time-dependent oscillator casts doubt on the importance of a known class of invariants. The existence of a Lagrangian function is shown to be inessential to the analysis.

1. Introduction

The simple harmonic oscillator

$$\ddot{x}+\omega^2 x=0 \tag{1.1}$$

represents a special case of the Klein-Gordon equation

$$(\Box-m^2)\phi=0 \tag{1.2}$$

where the spatial dimension has been reduced to zero. An (infinitesimal) 'symmetry' of (1.1) is defined here as any function $\lambda(x, \dot{x}, t)$ which satisfies $\ddot{\lambda}+\omega^2\lambda=0$ (see below). It was shown previously (Gordon 1981) that for non-zero space dimensions (and non-zero m) that the infinitesimal symmetries of (1.2) are restricted to linear functions in ϕ and its derivatives; correspondingly, conserved currents are essentially at most bilinear. For (1.1) this breaks down owing to the following: if $Q_1(x, \dot{x}, t)$ and $Q_2(x, \dot{x}, t)$ are invariants of (1.1) then so are each of the continuous infinity of functions which have the form

$$Q(x, \dot{x}, t)=F(Q_1, Q_2). \tag{1.3}$$

This 'functional composition' property distinguishes (1.1) from (1.2).

The interest here lies not so much in establishing the form of the invariants (obviously there are at most two functionally independent invariants here) but in the formalism used to associate invariants with symmetries.

In § 2 a generalised Noether theorem is described and applied to (1.1); it is shown that all invariants are generated (on an equal footing) in this way. In § 3 this is contrasted with the standard approach where a set of five 'preferred' invariants are generated. Sections 4 and 5 consider generalisations to (1.1), and § 6 contains a discussion and brief summary.

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2. Noether's theorem

2.1. Specialised formulation

The author's formulation of Noether's theorem (Gordon 1984) may be specialised to a single second-order ordinary differential equation

$$f(x, \dot{x}, \ddot{x}, t) = 0 \tag{2.1}$$

in the following way. Define the linearised operator $\tilde{f}$

$$\tilde{f}[\sigma] := \frac{\partial f}{\partial x}\sigma + \frac{\partial f}{\partial \dot{x}} D_t\sigma + \frac{\partial f}{\partial \ddot{x}} D_t^2\sigma \tag{2.2}$$

where $\sigma = \sigma(x, \dot{x}, t)$ is arbitrary and

$$D_t\sigma := \dot{x}\frac{\partial \sigma}{\partial x} + \ddot{x}\frac{\partial \sigma}{\partial \dot{x}} + \frac{\partial \sigma}{\partial t} \tag{2.3}$$

is the (total) time derivative operator. It is assumed that second- (and higher-) order derivatives are eliminated from (2.2) and (2.3) using (2.1). The adjoint operator $\tilde{f}^*$ is then

$$\tilde{f}^*[\sigma] := \frac{\partial f}{\partial x}\sigma - D_t\left(\frac{\partial f}{\partial \dot{x}}\sigma\right) + D_t^2\left(\frac{\partial f}{\partial \ddot{x}}\sigma\right).$$

Note that (2.1) may be derived from a Lagrangian function if and only if $\tilde{f}^* \equiv \tilde{f}$.

Let $Q(x, \dot{x}, t)$ be an invariant of (2.1), i.e. $D_tQ = 0$ or

$$D_tQ \equiv \lambda f \tag{2.4}$$

where the identity sign indicates that no use is made of (2.1) to eliminate $\ddot{x}$. $\lambda(x, \dot{x}, t)$ is the Lagrange multiplier which determines Q up to the addition of an arbitrary constant. The 'infinitesimal' form of Q, regarded as a linear operator, is

$$\tilde{Q}[\sigma] := (\partial Q/\partial x)\sigma + (\partial Q/\partial \dot{x})D_t\sigma. \tag{2.5}$$

Noether's theorem takes the following appearance:

$$D_tQ \equiv \lambda f \Rightarrow D_t\tilde{Q}[\sigma] = \lambda\tilde{f}[\sigma] \Leftrightarrow \tilde{f}^*[\lambda] = 0.$$

The first '$\Rightarrow$' follows from the linearisation of (2.4). This cannot normally be extended to '$\Leftrightarrow$' since an integrability condition is necessary for Q to exist. The second '$\Rightarrow$' states that the Lagrange multiplier satisfies the adjoint linearised equation. For a Lagrangian theory this is the same as the linearised equation itself and it is seen in § 3 that the standard Lie invariance arises as a special case. The '$\Leftarrow$' states that an 'infinitesimal' invariant exists for each solution of $\tilde{f}^*[\lambda] = 0$; it is easily verified that $\tilde{Q}$ may be defined by

$$\tilde{Q}[\sigma] = \lambda(\partial f/\partial \dot{x})\sigma + (\lambda\, \partial f/\partial \ddot{x})\overleftrightarrow{D}_t\sigma \tag{2.6}$$

where $\sigma\overleftrightarrow{D}_t\tau := \sigma D_t\tau - \tau D_t\sigma$. The integrability condition for $Q(x, \dot{x}, t)$ is easily found to be

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial f}{\partial \ddot{x}}\right) = \frac{\partial}{\partial \dot{x}}\left(\lambda\frac{\partial f}{\partial \dot{x}} - D_t\lambda\frac{\partial f}{\partial \ddot{x}}\right). \tag{2.7}$$

2.2. *Completeness*

Clearly there are at most two functionally independent invariants for (2.1), and correspondingly at most two linearly independent solutions to $\tilde{f}^*[\lambda]=0$. More precisely, if λ_1 and λ_2 are the Lagrange multipliers of Q_1 and Q_2 respectively, then from (1.3) the Lagrange multiplier of Q is

$$\lambda = (\partial F/\partial Q_1)\lambda_1 + (\partial F/\partial Q_2)\lambda_2.$$

Generally if λ_1 and λ_2 satisfy $\tilde{f}^*[\lambda]=0$, then so does

$$\lambda = Q_3\lambda_1 + Q_4\lambda_2 \tag{2.8}$$

where Q_3 and Q_4 are themselves invariants of (2.1).

2.3. *Simple harmonic oscillator*

Applying the above theory to (1.1), a Lagrange multiplier satisfies

$$\ddot{\lambda} + \omega^2\lambda = 0 \tag{2.9}$$

($\ddot{\lambda} \equiv D_t^2\lambda$). Equation (2.6) becomes

$$\tilde{Q}[\sigma] = \lambda\dot{\sigma} - \dot{\lambda}\sigma \tag{2.10}$$

and the integrability condition (2.7) reduces to

$$\partial\lambda/\partial x + \partial\dot{\lambda}/\partial\dot{x} = 0. \tag{2.11}$$

Two obvious solutions to (2.9) are $\lambda_1 = x_1(t)$, $\lambda_2 = x_2(t)$ where x_1 and x_2 are any independent pair of solutions of (1.1) (e.g. $x_1 = \sin\omega t$, $x_2 = \cos\omega t$). Then (2.11) is trivially satisfied and integration of (2.10) yields

$$Q_1 = x_1(t)\dot{x} - \dot{x}_1(t)x \qquad Q_2 = x_2(t)\dot{x} - \dot{x}_2(t)x. \tag{2.12}$$

From the above any third invariant is of the form (1.3).

It is interesting to seek solutions that are linear homogeneous in x and $\dot{x}$,

$$\lambda = \alpha(t)x + \beta(t)\dot{x}. \tag{2.13}$$

Equations (2.9) and (2.11) imply

$$\ddot{\alpha} = 2\omega^2\beta \qquad \dot{\beta} + 2\alpha = 0. \tag{2.14}$$

Therefore $\ddot{\alpha} + 4\omega^2\alpha = 0$ and a solution $\alpha = -2\mathrm{i}\omega\,\mathrm{e}^{2\mathrm{i}\omega t}$ is chosen. From (2.14), $\beta = 2\,\mathrm{e}^{2\mathrm{i}\omega t} + c$ where c is a constant and hence

$$\lambda = c\dot{x} + 2\,\mathrm{e}^{2\mathrm{i}\omega t}(\dot{x} - \mathrm{i}\omega x). \tag{2.15}$$

Integration of (2.10) then gives

$$Q = cE(x, \dot{x}) + Q'^2$$

where $E(x, \dot{x}) = \frac{1}{2}(\dot{x}^2 + \omega^2 x^2)$ is the energy and $Q' = \mathrm{e}^{\mathrm{i}\omega t}(\dot{x} - \mathrm{i}\omega x)$ is of the form (2.12). The only novel result obtained from (2.13) is the energy, which has the useful property of being independent of t.

The Lagrange multiplier for E is $\lambda = \dot{x}$.

3. Comparison with standard results

The standard theory of the Lie invariance of (1.1) was described by Wulfman and Wybourne (1976): under an infinitesimal transformation $x \to x + \delta x$, $t \to t + \delta t$

$$\delta x = \xi(x, t)\delta a \qquad \delta t = \eta(x, t)\delta a \tag{3.1}$$

where δa is an infinitesimal parameter, and

$$\mathscr{U} \equiv \xi\, \partial/\partial x + \eta\, \partial/\partial t \tag{3.2}$$

is the infinitesimal generator. $\mathscr{U}$ is extended to $\mathscr{U}''$ in a natural way to determine the change induced in any function $f(x, \dot{x}, \ddot{x}, t)$, and the condition for an infinitesimal transformation to be a symmetry of (1.1) is that

$$\mathscr{U}''(\ddot{x} + \omega^2 x) = 0. \tag{3.3}$$

In the present approach such a transformation would be considered to take place with t unchanged; this is not restrictive at all since (3.1) may then be replaced by

$$\bar{\delta} x = \delta x - \dot{x}\delta t \qquad \bar{\delta} t = 0$$

which yields the infinitesimal generator $\lambda\, \partial/\partial x$ with

$$\lambda = \xi(x, t) - \dot{x}\eta(x, t). \tag{3.4}$$

It is not hard to show that for (3.4), (2.9) is *identical* to (3.3). Therefore the present approach includes the standard one as a special case (i.e. when $\lambda(x, \dot{x}, t)$ is linear in $\dot{x}$).

Although the restriction that λ be of the form (3.4) may make sense from the geometrical point of view, it has no role to play in the association of invariants with symmetries. Indeed it is essentially the restriction that λ be linear in $\dot{x}$ that allows Lutsky (1978) to obtain a set of five invariants for (1.1); in view of the above results it is hard to believe that any special physical significance should be attached to these.

4. The time-dependent oscillator

In view of recent interest in the time-dependent oscillator

$$\ddot{x} + \Omega^2 x = 0$$

(Lewis 1968, Eliezer and Gray 1976, Prince and Eliezer 1980, Colegrave and Abdalla 1983) where $\Omega(t)$ is time-dependent, it is worth extending the present analysis. It is easily verified that (2.9)-(2.11) are unchanged, except that ω is replaced by $\Omega(t)$—this will be assumed in such equation references in this section.

It is not hard to show that for $\dot{\Omega}(t) \neq 0$, the only solution $\lambda(x, \dot{x})$ to (2.9) is $\lambda = x$; all other solutions include explicit time dependence. Since (2.11) does not hold, no finite invariant exists of the form $Q(x, \dot{x})$, and indeed this can also be verified directly from $\dot{Q} = 0$. In that case a complete solution to the problem of finding invariants is given by (taking arbitrary functions of) Q_1 and Q_2 of (2.12), but where $x_1(t)$ and $x_2(t)$ may not be explicitly written down without prior knowledge of the function $\Omega(t)$. Equation (2.12) may be inverted to give

$$x = (x_1(t)Q_2 - x_2(t)Q_1)/W \tag{4.1}$$

$$\dot{x} = (\dot{x}_1(t)Q_2 - \dot{x}_2(t)Q_1)/W \tag{4.2}$$

where $W := \dot{x}_1 x_2 - \dot{x}_2 x_1$ is the (non-zero, time-independent) Wronskian.

One may also seek solutions to (2.9) in the form (2.13),

$$\lambda = \alpha(t)x + \beta(t)\dot{x}$$

but as in § 2, any invariants generated will be functions of Q_1 and Q_2 (the results are interesting only for comparison with earlier work). Following the procedure set out in § 2, we obtain (after a suitable rescaling of β if necessary)

$$\ddot{\rho} + \Omega^2\rho = k\rho^{-3} \tag{4.3}$$

where $\beta = \rho^2$ (cf Prince and Eliezer 1980), $\alpha = -\rho\dot{\rho}$ and $k = 0$ or 1. Thus from (2.13)

$$\lambda = \rho^2\dot{x} - \rho\dot{\rho}x$$

which integrates via (2.10) to

$$Q = \tfrac{1}{2}(\rho\dot{x} - \dot{\rho}x)^2 + \tfrac{1}{2}k\bar{\rho}^2x^2(+\text{constant}). \tag{4.4}$$

If $k = 1$ then (4.4) is the invariant described by Lewis (1968) (see also Eliezer and Gray 1976, Prince and Eliezer 1980); this may be expressed in terms of Q_1 and Q_2 as follows. As is well known, any solution $\rho(t)$ of (4.3) ($k = 1$) may be expressed in the form

$$\rho^2 = Ax_1^2 + Bx_2^2 + 2Cx_1x_2 \tag{4.5}$$

where $x_1(t)$ and $x_2(t)$ are independent solutions of (1.1) and the constants A, B, C satisfy $AB - C^2 = W^{-2}$ (e.g. Prince and Eliezer 1980). Substitution of (4.1), (4.2) and (4.5) into (4.4) then yields (after some algebra)

$$Q_{\text{Lewis}} = \tfrac{1}{2}(AQ_1^2 + BQ_2^2 + 2CQ_1Q_2).$$

It is hard to believe that this particular quadratic function of Q_1 and Q_2 is of any more fundamental significance than (say) $Q = (Q_1 + Q_2)^4$. The use of (4.3) ($k = 1$) merely complicates a simple situation.

Once again it seems pointless to pick out special classes of invariant unless it is because some particular combination $Q = F(Q_1, Q_2)$ has some desirable property (such as the absence of explicit time dependence which is impossible here).

5. Non-Lagrangian oscillators

5.1. Damped oscillators

The damped oscillator

$$\ddot{x} + 2b\dot{x} + x = 0$$

(b = constant; $\omega = 1$ has been set by a suitable choice of time unit) does not possess a Lagrangian and conventionally no association between symmetries and invariants is made. The equation $\tilde{f}^*[\lambda] = 0$ becomes

$$\ddot{\lambda} - 2b\dot{\lambda} + \lambda = 0$$

which can be solved by

$$\lambda_\pm = \exp b_\pm t$$

where $b_\pm = b \pm (b^2 - 1)^{1/2}$. Equation (2.6) is now

$$\tilde{Q}[\sigma] = 2b\lambda\sigma + \lambda\dot{\sigma} - \sigma\dot{\lambda}$$

which integrates to give

$$Q_{\pm} = (2b\lambda_{\pm} - \dot{\lambda}_{\pm})x + \lambda_{\pm}\dot{x}.$$

Explicit time dependence may be eliminated by choosing the combination

$$Q = \frac{Q_{+}^{b_{-}}}{Q_{-}^{b_{+}}} = \frac{(\dot{x}+b_{-}x)^{b_{-}}}{(\dot{x}+b_{+}x)^{b_{+}}}$$

The absence of a Lagrangian has not hindered the analysis!

5.2. A nonlinear oscillator

The nonlinear equation

$$\ddot{x} + \dot{x}^2 x = 0$$

may be regarded as an oscillator which has a 'spring constant' equal to the square of the instantaneous velocity. $\tilde{f}^*[\lambda] = 0$ reduces to

$$\ddot{\lambda} - 2x\dot{x}\dot{\lambda} + (2x^2 - 1)\dot{x}^2\lambda = 0. \tag{5.1}$$

A solution is given by $\lambda = \dot{x}^{-1}$ which integrates via (2.6) to give

$$Q = \tfrac{1}{2}x^2 + \log \dot{x}.$$

Again the path from λ to Q is straightforward although it must be admitted that solving (5.1) (generally $\tilde{f}^*[\lambda] = 0$) may not always be as easy as it is here. Bearing (2.3) in mind, it is seen that (5.1) is a linear *partial* differential equation for $\lambda(x, \dot{x}, t)$ whose solution set is closed under taking linear combinations of the type (2.8).

6. Discussion and summary

Any reasonable Noether theorem associating symmetries with invariants should generate an uncountable infinity of invariants in order to give closure under functional composition. Although various definitions of a 'symmetry' are possible, in the present context this simple requirement is fundamental. It is worth noting that other authors have also identified the need for arbitrary functions in the context of the symmetries of the harmonic oscillator (e.g. Schwarz 1983).

Thus it has been argued that an (infinitesimal) symmetry should be defined as any solution $\sigma(x, \dot{x}, t)$ of $\tilde{f}[\sigma] = 0$. The standard case of invariance under a geometrical transformation in the (x, t) plane yields such a solution via its generator, as in (3.2) and (3.4). When $\tilde{f}$ is self-adjoint each symmetry yields at least an infinitesimal invariant, and closure under functional composition is guaranteed.

When $\tilde{f}$ is not self-adjoint, a symmetry does not immediately lead to an invariant, since it is solutions to $\tilde{f}^*[\lambda] = 0$ that are required. However, it has been shown that the analysis is not hindered provided that such solutions can be found.

A further possibility is that a fixed linear mapping $L: \sigma \to \lambda$ may be defined such that $\tilde{f}[\sigma] = 0 \Rightarrow \tilde{f}^*[\lambda] = 0$. This is indeed the case for Hamiltonian equations (Gordon 1984).

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References

Colegrave R K and Abdalla M S 1983 *J. Phys. A: Math. Gen.* **16** 3805-15
Eliezer C J and Gray A 1976 *SIAM J. Appl. Math.* **30** 463-8
Gordon T J 1981 *Math. Proc. Camb. Phil. Soc.* **90** 507-15
—— 1984 *Ann. Phys., NY* **155** 85-107
Lewis H R Jr 1968 *J. Math. Phys.* **9** 1976-86
Lutzky M 1978 *J. Phys. A: Math. Gen.* **11** 249-58
Prince G E and Eliezer C J 1980 *J. Phys. A: Math. Gen.* **13** 815-23
Schwarz F 1983 *J. Phys. A: Math. Gen.* **16** L133-5
Wulfman C E and Wybourne B G 1976 *J. Phys. A: Math. Gen.* **9** 507-18